

ABSTRACT CESÀRO SPACES: INTEGRAL REPRESENTATIONS

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ABSTRACT. The Cesàro function spaces $Ces_p = [\mathcal{C}, L^p]$, $1 \leq p \leq \infty$, have received renewed attention in recent years. Many properties of $[\mathcal{C}, L^p]$ are known. Less is known about $[\mathcal{C}, X]$ when the Cesàro operator takes its values in a rearrangement invariant (r.i.) space X other than L^p . In this paper we study the spaces $[\mathcal{C}, X]$ via the methods of vector measures and vector integration. These techniques allow us to identify the absolutely continuous part of $[\mathcal{C}, X]$ and the Fatou completion of $[\mathcal{C}, X]$; to show that $[\mathcal{C}, X]$ is never reflexive and never r.i.; to identify when $[\mathcal{C}, X]$ is weakly sequentially complete, when it is isomorphic to an AL-space, and when it has the Dunford-Pettis property. The same techniques are used to analyze the operator $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$; it is never compact but, it can be completely continuous.

INTRODUCTION

Cesàro function spaces have attracted much attention in recent times; see for example the papers [1], [2], [3] by Astashkin and Maligranda and [17], [18] by Lésnik and Maligranda and the references therein. These spaces arise when studying the behavior, in certain function spaces, of the Cesàro operator

$$\mathcal{C}: f \mapsto \mathcal{C}(f)(x) := \frac{1}{x} \int_0^x f(t) dt.$$

A classical result of Hardy motivated the study of the operator \mathcal{C} in the L^p spaces, thereby leading to the spaces $Ces_p := \{f : \mathcal{C}(|f|) \in L^p\}$. It was then natural to extend the investigation to the so called abstract Cesàro spaces $[\mathcal{C}, X]$, where the role of L^p is replaced by a more general function space X , namely, the Banach function space (B.f.s.)

$$[\mathcal{C}, X] := \{f : \mathcal{C}(|f|) \in X\},$$

equipped with the norm

$$\|f\|_{[\mathcal{C}, X]} := \|\mathcal{C}(|f|)\|_X, \quad f \in [\mathcal{C}, X].$$

We will focus our attention on those spaces X which are rearrangement invariant (r.i.) on $[0, 1]$.

It is known that $[\mathcal{C}, L^p] = Ces_p$ is not reflexive, [1, Theorem 1, Remark 1]. In Theorem 3.3 it is shown that $[\mathcal{C}, X]$ is never reflexive. This result is established via

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techniques from a different area. It turns out, for every r.i. space $X \neq L^\infty$, that the X -valued set function

$$m_X : A \mapsto m_X(A) := \mathcal{C}(\chi_A), \quad A \subseteq [0, 1] \text{ measurable},$$

is σ -additive, i.e., it is a *vector measure*. This fact can be successfully used for studying the function space $[\mathcal{C}, X]$. Indeed, the norm of $[\mathcal{C}, X]$ is not necessarily absolutely continuous (a.c.). Actually, the a.c. part $[\mathcal{C}, X]_a$ of $[\mathcal{C}, X]$ is precisely the well understood space $L^1(m_X)$ consisting of all the m_X -integrable functions (in the sense of Bartle, Dunford and Schwartz, [4]). Moreover, $[\mathcal{C}, X]$ need not have the Fatou property. It turns out that the Fatou completion $[\mathcal{C}, X]''$ of $[\mathcal{C}, X]$ is precisely the space $L_w^1(m_X)$ consisting of all the weakly m_X -integrable functions.

A further relevant point is that the integration operator $I_{m_X} : L^1(m_X) \rightarrow X$ given by $f \mapsto \int f dm_X$ is precisely the restriction to $[\mathcal{C}, X]_a$ of the Cesàro operator $\mathcal{C} : [\mathcal{C}, X] \rightarrow X$. Moreover, $L^1(m_X)$ is the *largest* B.f.s. over $[0, 1]$ with a.c. norm on which \mathcal{C} acts with values in X . In addition, the scalar variation measure $|m_X|$ of the vector measure m_X is always σ -finite and possesses a strongly measurable, Pettis integrable density $F : [0, 1] \rightarrow X$ relative to Lebesgue measure. A relevant feature for the operator \mathcal{C} (which a priori is only given by a pointwise expression on $[\mathcal{C}, X]$) is that *integral representations* become available. First, for \mathcal{C} restricted to $[\mathcal{C}, X]_a$, such a representation is given by

$$(1) \quad \mathcal{C}(f) = \int_{[0,1]} f dm_X, \quad f \in L^1(m_X) = [\mathcal{C}, X]_a,$$

via the Bartle-Dunford-Schwartz integral for vector measures. Actually, it turns out specifically for m_X that

$$(2) \quad \mathcal{C}(f) = \int_{[0,1]} f(y) F(y) dy, \quad f \in L^1(m_X),$$

which is defined more traditionally as a Pettis integral. Furthermore, for the class of r.i. spaces X where the variation measure $|m_X|$ is finite, the representation (2) when restricted to $L^1(|m_X|)$ is actually given via a *Bochner integrable density* F .

The paper is organized as follows.

In Section 1 we present the preliminaries on Banach function spaces, rearrangement invariant spaces and vector integration that are needed in the sequel.

Section 2 is devoted to establishing the main properties of the vector measure m_X . A large class of r.i. spaces X for which $|m_X|$ is a finite measure is identified; see Proposition 2.3 and Corollary 2.5.

In Section 3 the study of the space $[\mathcal{C}, X]$ is undertaken with the vector measure m_X and its space of integrable function $L^1(m_X)$ as main tools. As mentioned above, in Theorem 3.3 it is proved that $[\mathcal{C}, X]$ is never reflexive. It is also established as part of that result that $[\mathcal{C}, X]$ fails to be r.i. (this was proved for $[\mathcal{C}, L^p]$ in [2, Theorem 1] and conjectured in [17, Remark 3]). The problem of when $[\mathcal{C}, X]$ is order isomorphic to an AL-space, that is, to a Banach lattice where the norm is additive over disjoint functions, is also considered. It is shown (cf. Theorem 3.6(a)), for a large class of Lorentz spaces $\Lambda(\varphi)$, that $[\mathcal{C}, \Lambda(\varphi)]$ is order isomorphic to $L^1(|m_{\Lambda(\varphi)}|)$ with $|m_{\Lambda(\varphi)}|$ a finite, non-atomic measure. Crucial for the proof of the existence of this order

isomorphism is an identification, due to Lésnik and Maligranda, [17], of the associate space $[\mathcal{C}, X]'$ of the B.f.s. $[\mathcal{C}, X]$ (under some restrictions on the r.i. space X).

In Section 4 we analyze the operator $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$. The identification of the restriction of \mathcal{C} , via I_{m_X} , is used to show that the operator $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$ is never compact; see Proposition 4.2. For r.i. spaces X satisfying $X \subseteq L^1(|m_X|)$, which forces both m_X to have finite variation and $\mathcal{C}: X \rightarrow X$ to act boundedly, it follows (cf. Proposition 4.3) that $\mathcal{C}: X \rightarrow X$ is necessarily completely continuous. This result is quite useful in view of the fact that $\mathcal{C}: X \rightarrow X$ is never compact (whenever it is a bounded operator). The complete continuity of the restricted integration operator $I_{m_X}: L^1(|m_X|) \rightarrow X$ can be ‘lifted’ to the complete continuity of $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$, under some conditions on the r.i. space X ; see Proposition 4.3. This property of $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$ is related to $[\mathcal{C}, X]$ being order isomorphic to an AL-space; see Proposition 4.5. The section ends with another extension of a result valid for $X = L^p$. It was shown in [2, §6, Corollary 1] that the spaces Ces_p , $1 < p < \infty$, fail to have the Dunford-Pettis property. This result is extended to include all reflexive r.i. spaces X having a non-trivial upper Boyd index; see Proposition 4.7.

In the final section we discuss in fine detail the role of the Fatou property in relation to $[\mathcal{C}, X]$, and derive some consequences for $[\mathcal{C}, X]$; see Proposition 5.2.

We only consider r.i. spaces $X \neq L^\infty$ because $Ces_\infty = [\mathcal{C}, L^\infty]$, known as the Korenblyum-Kreĭn-Levin space, has already been thoroughly investigated; see [2], [3] and the references therein.

1. PRELIMINARIES

A *Banach function space* (B.f.s.) X on $[0,1]$ is a Banach space of classes of measurable functions on $[0,1]$ satisfying the ideal property, that is, $g \in X$ and $\|g\|_X \leq \|f\|_X$ whenever $f \in X$ and $|g| \leq |f|$ λ -a.e., where λ is the Lebesgue measure on $[0,1]$. The *associate space* X' of X consists of all functions g satisfying $\int_0^1 |f(t)g(t)| dt < \infty$, for every $f \in X$. The space X' is a subspace of the Banach space dual X^* of X . The *absolutely continuous* (a.c.) part X_a of X is the space of all functions $f \in X$ satisfying $\lim_{\lambda(A) \rightarrow 0} \|f\chi_A\|_X = 0$; here χ_A is the characteristic function of the set $A \in \mathcal{M}$, with \mathcal{M} denoting the σ -algebra of all Lebesgue measurable subsets of $[0,1]$. If $L^\infty \subseteq X_a$, then the closure of L^∞ in X coincides with X_a and $(X_a)' = X'$. The space X is said to have a.c. norm if $X = X_a$. In this case, $X' = X^*$. The space X satisfies the *Fatou property* if $\{f_n\} \subseteq X$ with $0 \leq f_n \leq f_{n+1} \uparrow f$ λ -a.e. and $\sup_n \|f_n\|_X < \infty$ imply that $f \in X$ and $\|f_n\|_X \rightarrow \|f\|_X$. The second associate space X'' of X is defined as $X'' = (X')'$. The space X has the Fatou property if and only if $X'' = X$. Unless specifically stated, it is not assumed that the Fatou property holds in X .

A *rearrangement invariant* (r.i.) space X on $[0,1]$ is a B.f.s. on $[0,1]$ such that if $g^* \leq f^*$ and $f \in X$, then $g \in X$ and $\|g\|_X \leq \|f\|_X$. Here f^* is the *decreasing rearrangement* of f , that is, the right continuous inverse of its distribution function: $\lambda_f(\tau) := \lambda(\{t \in [0,1] : |f(t)| > \tau\})$. The associate space X' of a r.i. space X is again a r.i. space. A r.i. space X satisfies $L^\infty \subseteq X \subseteq L^1$. If $X \neq L^\infty$, then $(X_a)' = X'$. The *fundamental function* φ_X of X is defined via $\varphi_X(t) := \|\chi_{[0,t]}\|_X$. For $X \neq L^\infty$ we have $\lim_{t \rightarrow 0} \varphi_X(t) = 0$, [26, Lemma 3, p.220].

Important classes of r.i. spaces are the Lorentz and Marcinkiewicz spaces. Let $\varphi: [0, 1] \rightarrow [0, \infty)$ be an increasing, concave function with $\varphi(0) = 0$. The Lorentz space $\Lambda(\varphi)$ consists of all measurable functions f on $[0, 1]$ satisfying

$$\|f\|_{\Lambda(\varphi)} := \int_0^1 f^*(s) d\varphi(s) < \infty.$$

Let $\varphi: [0, 1] \rightarrow [0, \infty)$ be a quasi-concave function, that is, φ is increasing, the function $t \mapsto \varphi(t)/t$ is decreasing and $\varphi(0) = 0$. The Marcinkiewicz space $M(\varphi)$ consists of all measurable functions f on $[0, 1]$ satisfying

$$\|f\|_{M(\varphi)} := \sup_{0 < t \leq 1} \frac{\varphi(t)}{t} \int_0^t f^*(s) ds < \infty.$$

The Marcinkiewicz space $M(\varphi)$ and the Lorentz space $\Lambda(\varphi)$ are, respectively, the largest and the smallest r.i. spaces having the fundamental function φ . That is, for any r.i. space X we have $\Lambda(\varphi_X) \subseteq X \subseteq M(\varphi_X)$. The associate space $\Lambda(\varphi)' = M(\psi)$ and $M(\varphi)' = \Lambda(\psi)$, for $\psi(t) := t/\varphi(t)$. In the notation of [15, p.144], observe that $M(\varphi) = M_\psi$.

If ϕ is a positive function defined on $[0, 1]$, then its lower and upper dilation indices are, respectively, defined by

$$\gamma_\phi := \lim_{t \rightarrow 0^+} \frac{\log \left(\sup_{0 < s \leq 1} \frac{\phi(st)}{\phi(s)} \right)}{\log t}, \quad \delta_\phi := \lim_{t \rightarrow +\infty} \frac{\log \left(\sup_{0 < s \leq 1/t} \frac{\phi(st)}{\phi(s)} \right)}{\log t}.$$

For a quasi-concave function φ it is known that $0 \leq \gamma_\varphi \leq \delta_\varphi \leq 1$. Whenever $\delta_\varphi < 1$ the following equivalence for the above norm in $M(\varphi)$ holds (see [15, Theorem II.5.3]):

$$(3) \quad \|f\|_{M(\varphi)} \asymp \sup_{0 < t \leq 1} \varphi(t) f^*(t).$$

The notation $A \asymp B$ means that there exist constants $C > 0$ and $c > 0$ such that $c \cdot A \leq B \leq C \cdot A$. For further details concerning r.i. spaces we refer to [5], [15], [19]; care should be taken with [5] as all r.i. spaces there are assumed to have the Fatou property. General references for B.f.s.' include [23], [29, Ch.15].

We recall briefly the theory of integration of real functions with respect to a vector measure, initially due to Bartle, Dunford and Schwartz, [4]. Let (Ω, Σ) be a measurable space, X a Banach space and $m: \Sigma \rightarrow X$ a σ -additive vector measure. For each $x^* \in X^*$, denote the \mathbb{R} -valued measure $A \mapsto \langle x^*, m(A) \rangle$ by x^*m and its variation measure by $|x^*m|$. A measurable function $f: \Omega \rightarrow \mathbb{R}$ is said to be *integrable with respect to m* if $f \in L^1(|x^*m|)$, for every $x^* \in X^*$, and for each $A \in \Sigma$ there exists a vector in X (denoted by $\int_A f dm$) satisfying $\langle \int_A f dm, x^* \rangle = \int_A f d x^*m$, for every $x^* \in X^*$. The m -integrable functions form a linear space in which

$$(4) \quad \|f\|_{L^1(m)} := \sup \left\{ \int |f| d|x^*m| : x^* \in X^*, \|x^*\| \leq 1 \right\}$$

is a seminorm. A set $A \in \Sigma$ is called *m -null* if $|x^*m|(A) = 0$ for every $x^* \in X^*$. Identifying functions which differ only in a m -null set, we obtain a Banach space

(of classes) of m -integrable functions, denoted by $L^1(m)$. It is a B.f.s. for the m -a.e. order and has a.c. norm. The simple functions are dense in $L^1(m)$ and the space $L^\infty(m)$ of all m -essentially bounded functions is contained in $L^1(m)$. The *integration operator* I_m from $L^1(m)$ to X is defined by $f \mapsto \int f dm := \int_\Omega f dm$. It is continuous, linear and has operator norm at most one. No assumptions have been made on the *variation measure* $|m|$ of m (cf. [23, §3.1]) in the definition of $L^1(m)$. In general $L^1(|m|) \subseteq L^1(m)$. We will repeatedly use the following property: let Y be the closed linear subspace of X generated by the range $m(\Sigma)$ of the vector measure m . Then $L^1(m_X) = L^1(m_Y)$ and $L^1(|m_X|) = L^1(|m_Y|)$, where $m_Y: \Sigma \rightarrow Y$ is given by $m_Y(A) := m_X(A)$ for all $A \in \Sigma$.

The B.f.s.' $L^1(m)$ can be quite different to the classical L^1 -spaces of scalar measures and may be difficult to identify explicitly. Indeed, every Banach lattice with a.c. norm and having a weak unit (e.g. $L^2([0, 1])$) is the L^1 -space of some vector measure, [8, Theorem 8]. For further details concerning $L^1(m)$ and I_m see, for example, [23, Ch.3] and the references therein.

2. THE VECTOR MEASURE INDUCED BY \mathcal{C}

The vector measure associated to the Cesàro operator is defined by

$$m: A \mapsto m(A) := \mathcal{C}(\chi_A), \quad A \in \mathcal{M}.$$

Since \mathcal{C} maps L^∞ into itself, we have $m(\mathcal{M}) \subseteq L^\infty$. So, m is a well defined, finitely additive vector measure with values in L^∞ but, it is not σ -additive as an L^∞ -valued measure, [25]. For every r.i. space X we have $L^\infty \subseteq X$. Accordingly, m is also well defined and finitely additive with values in X . We will denote m by m_X whenever it is necessary to indicate that the values of m are considered to be in X .

Theorem 2.1. *Let $X \neq L^\infty$ be a r.i. space.*

- (a) *The measure m_X is σ -additive.*
- (b) *The measure m_X has a strongly measurable, X -valued, Pettis λ -integrable density F given by*

$$(5) \quad F: y \in [0, 1] \mapsto F_y \in X \text{ with } F_y(x) =: \frac{1}{x} \chi_{[y, 1]}(x), \quad 0 < x \leq 1.$$

- (c) *The measure m_X has σ -finite variation given by*

$$(6) \quad |m_X|(A) = \int_A \|F_y\|_X dy, \quad A \in \mathcal{M}.$$

In the event that m_X has finite variation, F is actually Bochner λ -integrable.

- (d) *The range $m_X(\mathcal{M})$ of m_X is a relatively compact set in X .*

Proof. (a) Let (A_n) be a sequence of sets with $A_n \downarrow \emptyset$. Then the functions (χ_{A_n}) decrease pointwise to zero. Since \mathcal{C} is a positive operator, the sequence $(\mathcal{C}(\chi_{A_n}))$ is also decreasing; by the Dominated Convergence Theorem applied to $\chi_{A_n} \downarrow 0$ it follows that $(\mathcal{C}(\chi_{A_n}))$ actually decreases to zero a.e. Recall that $m_X(\mathcal{M}) \subseteq L^\infty \subseteq X_a$. But, X_a has a.c. norm and so $\|\mathcal{C}(\chi_{A_n})\|_{X_a} \rightarrow 0$. Since the norms of X_a and X coincide, we have $\|\mathcal{C}(\chi_{A_n})\|_X \rightarrow 0$, i.e., $m_X(A_n) \rightarrow 0$ in X .

(b) Consider the X -valued vector function F given by (5). It is a.e. well defined since, for each $0 < y \leq 1$, we have $F_y \in L^\infty \subset X$. To prove that it is strongly measurable it suffices to verify that $y \in (0, 1] \mapsto F_y \in X$ is continuous. Fix $0 < t < s \leq 1$, in which case

$$\|F_t - F_s\|_X = \left\| \frac{1}{x} \chi_{[t,s]} \right\|_X \leq \frac{1}{t} \varphi_X(s - t).$$

Since $X \neq L^\infty$, it follows that $\varphi_X(s - t) \rightarrow 0$ as $(s - t) \rightarrow 0$.

Next we check the Pettis λ -integrability of F . Note that $F_y \in X_a$ for $y \in (0, 1]$. For any $0 \leq g \in (X_a)'$ we have, via Fubini's theorem, that

$$\int_0^1 \langle F_y, g \rangle dy = \int_0^1 \int_0^1 \frac{1}{x} \chi_{[y,1]}(x) g(x) dx dy = \int_0^1 g(x) dx,$$

which is surely finite as $(X_a)' \subseteq L^1$. Since elements of X^* restricted to X_a belong to $(X_a)^*$ and $(X_a)^* = (X_a)'$, it follows that $y \mapsto \langle F_y, x^* \rangle \in L^1$ for every $x^* \in X^*$.

It remains to check that F is the Pettis λ -integrable density for m_X . Fix $A \in \mathcal{M}$ and recall that $m_X(A) = \mathcal{C}(\chi_A) \in X_a$. For $0 \leq g \in (X_a)'$ an application of Fubini's theorem yields

$$\begin{aligned} \langle m_X(A), g \rangle &= \int_0^1 g(x) \left(\int_0^1 \frac{1}{x} \chi_{[0,x]}(y) \chi_A(y) dy \right) dx \\ &= \int_A \int_0^1 g(x) \frac{1}{x} \chi_{[y,1]}(x) dx dy \\ &= \int_A \left\langle \frac{1}{x} \chi_{[y,1]}(x), g(x) \right\rangle dy \\ &= \int_A \langle F_y, g \rangle dy. \end{aligned}$$

Since this is valid for every $0 \leq g \in (X_a)'$ and $(X_a)^* = (X_a)'$, it follows that $F: y \mapsto F_y$ is Pettis λ -integrable with

$$(7) \quad \int_A F_y dy := m_X(A) \in X_a \subset X, \quad A \in \mathcal{M}.$$

(c) Fix $0 < a < 1$. Consider the measure m_X restricted to $[a, 1]$. Since F is continuous on the compact set $[a, 1]$, we have $\int_{[a,1]} \|F_y\|_X dy < \infty$. According to (7), $y \mapsto F_y$ is then a Bochner λ -integrable density for m_X on $[a, 1]$. Accordingly,

$$(8) \quad |m_X|(A) = \int_A \|F_y\|_X dy, \quad A \in \mathcal{M}, \quad A \subseteq [a, 1].$$

Let now $A \in \mathcal{M}$. Set $A_n := [1/n, 1] \cap A$, for $n \geq 2$, in which case $|m_X|(A_n) < \infty$. Observing that $\chi_{A_n}(y) \|F_y\|_X \uparrow \chi_A(y) \|F_y\|_X$ λ -a.e. it follows from (8) and the σ -additivity of $|m_X|$ that

$$m_X(A) = \lim_n m_X(A_n) = \lim_n \int_{A_n} \|F_y\|_X dy = \int_A \|F_y\|_X dy.$$

This establishes (6) and the σ -finiteness of the variation.

In the event that m_X has finite variation, (6) implies that $y \mapsto \|F_y\|_X$ belongs to L^1 and hence F , being strongly measurable, is Bochner λ -integrable.

(d) Set $D_n = (1/2^n, 1/2^{n-1}]$, for $n \geq 1$. Then for each $n \geq 1$ we have $|m_X|(D_n) < \infty$. Moreover, the density F is Bochner λ -integrable over each D_n . Hence, the range $m_X(\mathcal{M}_{D_n})$ is relatively compact in X , [23, p.148], where $\mathcal{M}_{D_n} := \{A \in \mathcal{M} : A \subseteq D_n\}$. Thus,

$$m_X(\mathcal{M}) = \sum_{n=1}^{\infty} m_X(\mathcal{M}_{D_n}) := \left\{ \sum_{n=1}^{\infty} f_n : f_n \in m_X(\mathcal{M}_{D_n}) \text{ for } n \in \mathbb{N} \right\}.$$

Arguing as in the proof of Corollary 2.43 (see also part II of Proposition 3.56) in [23] we deduce that $m_X(\mathcal{M})$ is relatively compact in X . \square

Remark 2.2. (a) The σ -finiteness of $|m_X|$ also follows from a general result on Pettis integration, [28, Proposition 5.6(iv)]. Since $([0, 1], \mathcal{M}, \lambda)$ is a perfect measure space, the relative compactness of $m_X(\mathcal{M})$ in X is also a general result (due to C. Stegall), [28, Proposition 5.7].

(b) It follows from (6) that λ and m_X have the same null sets.

For certain r.i. spaces X it is possible to compute $|m_X|$ precisely.

Proposition 2.3. *For the Lorentz space $\Lambda(\varphi)$ we have*

$$(9) \quad \|F_y\|_{\Lambda(\varphi)} = \int_0^{1-y} \frac{\varphi'(t)}{t+y} dt, \quad y \in (0, 1],$$

and

$$|m_{\Lambda(\varphi)}|([0, 1]) = \int_0^1 \log(1/t) \varphi'(t) dt.$$

Consequently, $m_{\Lambda(\varphi)}$ has finite variation precisely when $\log(1/t) \in \Lambda(\varphi)$ and, in that case, $|m_{\Lambda(\varphi)}|([0, 1]) = \|\log(1/t)\|_{\Lambda(\varphi)}$.

Proof. For $y \in (0, 1]$ the decreasing rearrangement of $F_y(\cdot)$ is given by

$$(10) \quad (F_y)^*(t) = F_y(t+y) = \frac{1}{t+y} \chi_{[0, 1-y]}(t), \quad 0 \leq t \leq 1.$$

It follows that

$$\|F_y\|_{\Lambda(\varphi)} = \int_0^{1-y} \frac{\varphi'(t)}{t+y} dt, \quad y \in (0, 1].$$

Then, from (6) we can conclude that

$$|m_{\Lambda(\varphi)}|(A) = \int_A \|F_y\|_{\Lambda(\varphi)} dy = \int_A \left(\int_0^{1-y} \frac{\varphi'(t)}{t+y} dt \right) dy.$$

For $A = [0, 1]$ an application of Fubini's theorem yields

$$|m_{\Lambda(\varphi)}|([0, 1]) = \int_0^1 \left(\int_0^{1-y} \frac{\varphi'(t)}{t+y} dt \right) dy = \int_0^1 \log(1/t) \varphi'(t) dt.$$

Since $t \mapsto \log(1/t)$ is decreasing, it is clear that $m_{\Lambda(\varphi)}$ has finite variation precisely when $\log(1/t) \in \Lambda(\varphi)$ in which case $|m_{\Lambda(\varphi)}|([0, 1]) = \|\log(1/t)\|_{\Lambda(\varphi)}$. \square

Example 2.4. The Zygmund spaces of exponential integrability L_{exp}^p , for $p > 0$, are “close” to L^∞ ; see [5, Definition IV.6.11]. The classical space L_{exp} (i.e. $p = 1$) is a particular case. The space L_{exp}^p coincides with the Marcinkiewicz space $M(\varphi_p)$ for $\varphi_p(t) := \log^{-1/p}(e/t)$. For $X = \Lambda(\varphi_p)$ we have that $\log(1/t) \in \Lambda(\varphi_p)$ if and only if $0 < p < 1$. Hence, in view of Proposition 2.3, $m_{\Lambda(\varphi_p)}$ has finite variation if and only if $0 < p < 1$.

Let X, Y be r.i. spaces with $X \subseteq Y$, in which case there exists $K > 0$ such that $\|f\|_Y \leq K\|f\|_X$ for $f \in X$. In particular, $\|m_Y(A)\|_Y \leq K\|m_X(A)\|_X$ for $A \in \mathcal{M}$. Hence, m_Y has finite variation whenever m_X does. This observation, together with Proposition 2.3 and Example 2.4 establishes the following result.

Corollary 2.5. *Let $X \neq L^\infty$ be a r.i. space. Suppose that $\Lambda(\varphi) \subseteq X$ for some increasing, concave function φ satisfying $\varphi(0) = 0$ and*

$$\int_0^1 \log(1/t) \varphi'(t) dt < \infty,$$

that is, $\log(1/t) \in \Lambda(\varphi)$. Then m_X has finite variation.

In particular, since $\Lambda(\varphi_p) \subseteq M(\varphi_p) = L_{\text{exp}}^p$, this is the case if $L_{\text{exp}}^p \subseteq X$ for some $0 < p < 1$.

Example 2.6. According to Corollary 2.5, m_X has finite variation whenever X is a Lorentz space $L^{p,q}$ on $[0, 1]$ for $(p, q) \in (1, \infty) \times [1, \infty]$ or for $p = q = 1$ (see [5, Definition IV.4.1]), and whenever X is an Orlicz space L^Φ satisfying $\Phi(t) \leq e^{tp}$, $t \geq t_0$, for some $p \in (0, 1)$.

3. THE CESÀRO SPACE $[\mathcal{C}, X]$

In [13] a study of optimal domains for kernel operators $Tf(x) = \int_0^1 f(y)K(x, y) dy$ was undertaken. Although the conditions imposed on the kernel $K(x, y)$ in [13, §3] do not apply to the kernel $(x, y) \mapsto (1/x)\chi_{[0,x]}(y)$ generating the Cesàro operator, a detailed analysis of the arguments given there shows that the only condition needed for the results to remain valid for r.i. spaces $X \neq L^\infty$ is that the partial function $K_x: y \mapsto K(x, y)$ belongs to L^1 for a.e. $x \in [0, 1]$. The remaining conditions were aimed purely at guaranteeing that the vector measure associated with the kernel was σ -additive as an L^∞ -valued measure which, in turn, was the way of ensuring the σ -additivity of the measure when interpreted as an X -valued measure (for every r.i. space $X \neq L^\infty$). This last condition of σ -additivity is obtained, for the case when T is the Cesàro operator, by other means; see Theorem 2.1(a). Accordingly, from the results of §3 of [13] we have the following facts.

Proposition 3.1. *Let $X \neq L^\infty$ be a r.i. space. The following assertions hold.*

- (a) *If $f \in L^1(m_X)$, then $f \in [\mathcal{C}, X]$ and $\|f\|_{L^1(m_X)} = \|f\|_{[\mathcal{C}, X]}$.*
- (b) *If X has a.c. norm, then $[\mathcal{C}, X]$ has a.c. norm and $[\mathcal{C}, X] = L^1(m_X)$.*
- (c) $[\mathcal{C}, X_a] = [\mathcal{C}, X]_a$.

Consequently, the following chain of inclusions holds

$$(11) \quad L^1(|m_X|) \subseteq L^1(m_X) = L^1(m_{X_a}) = [\mathcal{C}, X_a] = [\mathcal{C}, X]_a \subseteq [\mathcal{C}, X].$$

In this section we will study various properties of $[\mathcal{C}, X]$ and examine certain connections between the spaces appearing in (11).

The containment $L^1(m_X) \subseteq [\mathcal{C}, X]$ can be strict, as seen by the following result.

Proposition 3.2. *Let φ be an increasing, concave function with $\varphi(0) = 0$ and upper dilation index $\delta_\varphi < 1$. For the corresponding Marcinkiewicz space $M(\varphi)$ the containment $L^1(m_{M(\varphi)}) \subseteq [\mathcal{C}, M(\varphi)]$ is strict.*

Proof. The a.c.-part of the space $M(\varphi)$ is

$$M(\varphi)_a = M(\varphi)_0 := \left\{ f : \lim_{t \rightarrow 0} \frac{\varphi(t)}{t} \int_0^t f^*(s) ds = 0 \right\}.$$

The condition $\delta_\varphi < 1$ allows us to use the equivalent expression for the norm in $M(\varphi)$ given by (3). The function $1/\varphi$ is decreasing and so $(1/\varphi)^* = 1/\varphi$. It follows that $\|1/\varphi\|_{M(\varphi)} \asymp 1$ and hence, $1/\varphi \in M(\varphi)$. On the other hand,

$$\frac{\varphi(t)}{t} \int_0^t \left(\frac{1}{\varphi} \right)^* (s) ds \geq \frac{\varphi(t)}{t} \frac{t}{\varphi(t)} = 1, \quad t \in (0, 1],$$

showing that $1/\varphi \notin M(\varphi)_0$. So, $1/\varphi \in M(\varphi) \setminus M(\varphi)_0$.

To verify that $\mathcal{C}(1/\varphi) \asymp 1/\varphi$ is equivalent to showing that $(\varphi(t)/t) \int_0^t ds/\varphi(s) \asymp 1$. Since $1/\varphi$ is decreasing (i.e., $(1/\varphi)^* = 1/\varphi$), this is equivalent to verifying $\|1/\varphi\|_{M(\varphi)} \asymp 1$, that is, to showing that $1/\varphi \in M(\varphi)$. But, we have just proved that this is indeed the case, due to the condition $\delta_\varphi < 1$. Hence, $\mathcal{C}(1/\varphi) \in M(\varphi) \setminus M(\varphi)_0$ which implies that $1/\varphi \in [\mathcal{C}, M(\varphi)] \setminus [\mathcal{C}, M(\varphi)_0]$. From (11) we have that $L^1(m_{M(\varphi)}) = L^1(m_{M(\varphi)_0}) = [\mathcal{C}, M(\varphi)_0]$. Consequently, $1/\varphi \in [\mathcal{C}, M(\varphi)] \setminus L^1(m_{M(\varphi)})$. \square

We now establish two properties of $[\mathcal{C}, X]$ that were alluded to in the Introduction.

Theorem 3.3. *Let $X \neq L^\infty$ be any r.i. space.*

- (a) *The space $L^1(m_X)$ is not reflexive. Hence, the Cesàro space $[\mathcal{C}, X]$ is not reflexive either.*
- (b) *The Cesàro space $[\mathcal{C}, X]$ is not r.i. Moreover, neither is $L^1(m_X) = [\mathcal{C}, X]_a$.*

Proof. (a) A general result concerning the L^1 -space of a vector measure m asserts that if m has σ -finite variation and no atoms, then $L^1(m)$ is not reflexive, [10, Remark p.3804], [23, Corollary 3.23(ii)]. Since this is the case for $L^1(m_X)$, which is a closed subspace of $[\mathcal{C}, X]$, it follows that $[\mathcal{C}, X]$ is not reflexive either.

(b) Let $\varphi := \varphi_X$ be the fundamental function of X . Set $f(t) := (-2\varphi^{-1/2})'(1-t) = \varphi'(1-t)/\varphi^{3/2}(1-t)$. Since f is an increasing function, $f^*(t) = \varphi'(t)/\varphi^{3/2}(t)$. Direct computation shows that $\mathcal{C}f^* \equiv \infty$. Thus, $f^* \notin [\mathcal{C}, X]$.

On the other hand,

$$\begin{aligned}
\mathcal{C}f(x) &= \frac{1}{x} \int_0^x \frac{\varphi'(1-t)}{\varphi^{3/2}(1-t)} dt = \frac{1}{x} \int_{1-x}^1 \frac{\varphi'(s)}{\varphi^{3/2}(s)} ds \\
&= \frac{2}{x} \left(\frac{1}{\varphi^{1/2}(1-x)} - \frac{1}{\varphi^{1/2}(1)} \right) \\
&= \frac{2}{\varphi^{1/2}(1)} \left(\frac{\varphi^{1/2}(1) - \varphi^{1/2}(1-x)}{x} \right) \frac{1}{\varphi^{1/2}(1-x)} \\
&:= \frac{h(x)}{\varphi^{1/2}(1-x)}.
\end{aligned}$$

Both of the functions h and $1/h$ are bounded on $[0, 1]$. Accordingly, $\mathcal{C}(f)$ is equivalent to $x \mapsto 1/\varphi^{1/2}(1-x)$. So, $(\mathcal{C}f)^*(t) \asymp 1/\varphi^{1/2}(t)$. It follows that

$$\|\mathcal{C}f\|_{\Lambda(\varphi)} \asymp \int_0^1 \frac{1}{\varphi^{1/2}(t)} \varphi'(t) dt = 2\varphi^{1/2}(1) < \infty.$$

Hence, $(\mathcal{C}f)^* \in \Lambda(\varphi) \subseteq X$, which implies that $f \in [\mathcal{C}, X]$. So, $[\mathcal{C}, X]$ is not r.i.

According to (11) we have $L^1(m_{X_a}) = [\mathcal{C}, X_a]$. Since $[\mathcal{C}, X_a]$ fails to be r.i., so does $L^1(m_{X_a})$. But, $L^1(m_{X_a}) = L^1(m_X)$. Accordingly, the closed subspace $L^1(m_X)$ of $[\mathcal{C}, X]$ is never r.i. \square

Remark 3.4. (a) A reasonable ‘substitute’ for reflexivity is weak sequential completeness. If X is weakly sequentially complete, then $[\mathcal{C}, X]$ is also weakly sequentially complete. Indeed, the weak sequential completeness of X implies that of $L^1(m_X)$, [8, Corollary to Theorem 3]. But, $L^1(m_X) = [\mathcal{C}, X]$; see Proposition 3.1(a).

(b) Some further examples of vector measures m for which the spaces $L^1(m)$ are known not to be r.i. arise from Rademacher functions, [11, Theorem 1], and from fractional integrals, [12, Example 5.15(b)].

We now address the question of when $[\mathcal{C}, X]$ is order isomorphic to an *AL-space*, that is, to a Banach lattice in which the norm is additive over disjoint functions. In this regard, the space $X = L^1$ exhibits a particular feature, namely, that

$$(12) \quad [\mathcal{C}, L^1] = L^1(m_{L^1}) = L^1(|m_{L^1}|) = L^1(\log(1/t)).$$

We point out that not only do the three spaces $[\mathcal{C}, L^1]$, $L^1(m_{L^1})$ and $L^1(|m_{L^1}|)$ coincide, but that $[\mathcal{C}, L^1]$ is also an AL-space.

Proposition 3.5. *Let $X \neq L^\infty$ be a r.i. space. The following conditions are equivalent.*

- (a) *The space $[\mathcal{C}, X]$ is order isomorphic to an AL-space.*
- (b) *The spaces $L^1(m_X)$ and $L^1(|m_X|)$ are order isomorphic via the natural inclusion (this latter condition is written as $L^1(m_X) \simeq L^1(|m_X|)$).*
- (c) *The function $y \mapsto \|F_y\|_X$, $y \in [0, 1]$, belongs to the associate space $[\mathcal{C}, X]'$.*

If any one of these conditions holds, then

$$[\mathcal{C}, X] = L^1(m_X) \simeq L^1(|m_X|).$$

Proof. (a) \Rightarrow (b) If $[\mathcal{C}, X]$ is order isomorphic to an AL-space, then it is a.c., [19, Theorem 1.a.5 and Proposition 1.a.7]. Hence, by (11) we have that $[\mathcal{C}, X] = [\mathcal{C}, X]_a = L^1(m_X)$ and so $L^1(m_X)$ is order isomorphic to an AL-space. This last condition implies that $L^1(m_X)$ is order isomorphic (via the natural inclusion) to $L^1(|m_X|)$; see Proposition 2 of [9] and its proof.

(b) \Rightarrow (a) Suppose that $L^1(m_X) \simeq L^1(|m_X|)$. According to (11) we have $[\mathcal{C}, X]_a = L^1(m_X)$ and so $[\mathcal{C}, X]_a \simeq L^1(|m_X|)$. Since $L^1(|m_X|)$ is weakly sequentially complete, it follows that $[\mathcal{C}, X]_a$ has the Fatou property and hence, that $[\mathcal{C}, X]_a = ([\mathcal{C}, X]_a)''$. Since $[\mathcal{C}, X]_a \neq \{0\}$, we have $([\mathcal{C}, X]_a)' = [\mathcal{C}, X]'$ and hence, $([\mathcal{C}, X]_a)'' = [\mathcal{C}, X]''$. Thus, $[\mathcal{C}, X]_a = [\mathcal{C}, X]''$ which, together with the chain of inclusions $[\mathcal{C}, X]_a \subseteq [\mathcal{C}, X] \subseteq [\mathcal{C}, X]''$, yields $[\mathcal{C}, X] = [\mathcal{C}, X]_a$. Accordingly, $[\mathcal{C}, X] \simeq L^1(|m_X|)$ and this last space is an AL-space.

(b) \Leftrightarrow (c) Due to (11) we have $L^1(m_X) = [\mathcal{C}, X_a]$. Hence, the condition $L^1(m_X) \simeq L^1(|m_X|)$ is equivalent to $[\mathcal{C}, X_a] \simeq L^1(|m_X|)$. This, in turn, is equivalent to the requirement

$$\int_0^1 |f(y)| \cdot \|F_y\|_X dy < \infty, \quad f \in [\mathcal{C}, X_a],$$

which is precisely the condition that the function $y \mapsto \|F_y\|_X$ belongs to the associate space $[\mathcal{C}, X_a]' = [\mathcal{C}, X]'$. \square

In the sequel we will repeatedly use the fact that $\mathcal{C}: X \rightarrow X$ (necessarily boundedly) if and only if $X \subseteq [\mathcal{C}, X]$. For r.i. spaces X this corresponds precisely to the upper Boyd index $\bar{\alpha}_X$ of X satisfying $\bar{\alpha}_X < 1$; see [15, II.6.7, Theorem 6.6] or [20, Remark 5.13]. Note that the proof given in [5, Theorem III.5.15] uses the Fatou property of X . Observe that if $\mathcal{C}: X \rightarrow X$, then also $\mathcal{C}: X_a \rightarrow X_a$.

Theorem 3.6. *Let φ be an increasing, concave function with $\varphi(0) = 0$ and having non-trivial dilation indices $0 < \gamma_\varphi \leq \delta_\varphi < 1$.*

- (a) *For $X = \Lambda(\varphi)$ the B.f.s. $[\mathcal{C}, X]$ is order isomorphic to an AL-space.*
- (b) *For $X = M(\varphi)$ the B.f.s. $[\mathcal{C}, X]$ is not order isomorphic to an AL-space.*

Proof. Via Proposition 3.5, we need to decide whether or not $L^1(m_X) \simeq L^1(|m_X|)$.

In [17, Corollary 13] Lesník and Maligranda identify the associate space of $[\mathcal{C}, X]$ in the case when X has the Fatou property and both $\mathcal{C}, \mathcal{C}^*$ act boundedly on X . Here $f \mapsto \mathcal{C}^*(f)(x) := \int_x^1 \frac{f(t)}{t} dt$, $x \in [0, 1]$, for any a.e. finite measurable function f (denoted by $f \in L^0$) for which it is meaningfully defined, is the *Copson operator*. Then

$$(13) \quad [\mathcal{C}, X]' = \left(X' \left(\frac{1}{1-x} \right) \right)^\sim = \left\{ f : y \mapsto \frac{\tilde{f}(y)}{1-y} \in X' \right\},$$

where \tilde{f} is the decreasing majorant of f , defined by $\tilde{f}(y) := \sup_{x \geq y} |f(x)|$ and, for a weight function $0 < w$ on $[0, 1]$ and a B.f.s. Y , we set $Y(w) := \{h : wh \in Y\}$ and $\tilde{Y} := \{g : \tilde{g} \in Y\}$.

(a) The identification (13) applies to $X = \Lambda(\varphi)$ as X possesses the Fatou property and because $\underline{\alpha}_X = \gamma_\varphi$ and $\bar{\alpha}_X = \delta_\varphi$, together with the given index assumptions,

imply that $0 < \underline{\alpha}_X \leq \overline{\alpha}_X < 1$ which, in turn, guarantees that $\mathcal{C}, \mathcal{C}^*: \Lambda(\varphi) \rightarrow \Lambda(\varphi)$ boundedly.

Since $\Lambda(\varphi)' = M(\psi)$, for $\psi(t) := t/\varphi(t)$, we have from (13) that

$$[\mathcal{C}, \Lambda(\varphi)]' = \left(M(\psi) \left(\frac{1}{1-y} \right) \right)^{\sim} = \left\{ f : \sup_{0 < t \leq 1} \frac{1}{\varphi(t)} \int_0^t \left(\frac{\tilde{f}(y)}{1-y} \right)^*(s) ds < \infty \right\}.$$

The condition $0 < \gamma_\varphi$ implies that $\delta_\psi < 1$ which allows us, via (3), to simplify the previous description to

$$[\mathcal{C}, \Lambda(\varphi)]' = \left\{ f : \sup_{0 < t \leq 1} \frac{t}{\varphi(t)} \left(\frac{\tilde{f}(y)}{1-y} \right)^*(t) < \infty \right\}.$$

We need to verify that $y \mapsto \|F_y\|_{\Lambda(\varphi)} \in [\mathcal{C}, \Lambda(\varphi)]'$; see Proposition 3.5. From (9) it follows that

$$\|F_y\|_{\Lambda(\varphi)} = \int_0^{1-y} \frac{\varphi'(s)}{y+s} ds.$$

This function is decreasing (as a function of its variable y), so it coincides with its decreasing majorant, that is, $(\|F_y\|_{\Lambda(\varphi)})^{\sim} = \|F_y\|_{\Lambda(\varphi)}$. Moreover, for $0 < y \leq 1$, we have

$$\begin{aligned} \frac{\|F_y\|_{\Lambda(\varphi)}}{1-y} &\leq 2\chi_{[0,1/2]}(y) \int_0^1 \frac{\varphi'(s)}{y+s} ds + \chi_{[1/2,1]}(y) \frac{2}{1-y} \int_0^{1-y} \varphi'(s) ds \\ &\leq 2 \int_0^1 \frac{\varphi'(s)}{y+s} ds + 2 \frac{\varphi(1-y)}{1-y} \\ &:= g(y) + h(y). \end{aligned}$$

In the latter term, g is decreasing and h is increasing due to the quasi-concavity of φ (which implies that $\varphi(t)/t$ is decreasing), i.e., $g^* = g$ and $h^*(t) = h(1-t)$. Using the property $(g+h)^*(t) \leq g^*(t/2) + h^*(t/2)$ (see (2.23) in [15, Ch.II §2, p.67]), it follows that

$$\left(\frac{\|F_y\|_{\Lambda(\varphi)}}{1-y} \right)^*(t) \leq g\left(\frac{t}{2}\right) + h\left(1 - \frac{t}{2}\right) = 2 \int_0^1 \frac{\varphi'(s)}{\frac{t}{2} + s} ds + 2 \frac{\varphi(t/2)}{t/2}.$$

Accordingly,

$$\sup_{0 < t \leq 1} \frac{t}{\varphi(t)} \left(\frac{\|F_y\|_{\Lambda(\varphi)}}{1-y} \right)^*(t) \leq 2 \sup_{0 < t \leq 1} \frac{t}{\varphi(t)} \int_0^1 \frac{\varphi'(s)}{\frac{t}{2} + s} ds + 4 \sup_{0 < t \leq 1} \frac{t}{\varphi(t)} \frac{\varphi(t/2)}{t}.$$

The last term in the right-side is bounded (as φ increasing implies $\varphi(t/2)/\varphi(t) \leq 1$) and so we concentrate on the first term. Due to the quasi-concavity of φ we have $t\varphi'(t) \leq \varphi(t)$. This, together with a change of variables yields, for $t \in (0, 1]$, that

$$\frac{t}{\varphi(t)} \int_0^1 \frac{\varphi'(s)}{\frac{t}{2} + s} ds \leq \int_0^1 \frac{\varphi(s)}{\varphi(t)} \frac{t}{s(\frac{t}{2} + s)} ds \leq \int_0^{1/t} \frac{\varphi(tu)}{\varphi(t)} \frac{2du}{u(1+u)} = I_t.$$

The conditions $0 < \gamma_\varphi \leq \delta_\varphi < 1$ imply that there exist $\alpha, \beta \in (0, 1)$, and u_0, u_1 with $0 < u_0 < 1 < u_1 < \infty$ such that

$$\frac{\varphi(tu)}{\varphi(t)} \leq u^\alpha, \quad 0 < u < u_0, \quad \frac{\varphi(tu)}{\varphi(t)} \leq u^\beta, \quad u > u_1,$$

[15, pp.53-56]. Since $\varphi(tu)/\varphi(t) \leq \max\{1, u_1\} = u_1$, for $u_0 < u < u_1$ (via the quasi-concavity of φ), it follows that

$$I_t \leq \int_0^{u_0} \frac{2u^\alpha du}{u(1+u)} + \int_{u_0}^{u_1} u_1 \frac{2du}{u(1+u)} + \int_{u_1}^\infty \frac{2u^\beta du}{u(1+u)},$$

which is finite as $0 < \alpha, \beta < 1$. Thus, $\|F_t\|_{\Lambda(\varphi)} \in [\mathcal{C}, \Lambda(\varphi)]'$ and so $L^1(m_{\Lambda(\varphi)}) \simeq L^1(|m_{\Lambda(\varphi)}|)$. Hence, $[\mathcal{C}, \Lambda(\varphi)]$ is order isomorphic to an AL-space.

(b) For $X = M(\varphi)$ the identification (13) can again be applied, for the same reasons that it was applied in the case of $\Lambda(\varphi)$; see part (a). In particular, both $\mathcal{C}, \mathcal{C}^*: M(\varphi) \rightarrow M(\varphi)$ boundedly.

Since $M(\varphi)' = \Lambda(\psi)$, for $\psi(t) := t/\varphi(t)$, we have from (13) that

$$[\mathcal{C}, M(\varphi)]' = \left(\Lambda(\psi) \left(\frac{1}{1-y} \right) \right)^{\sim} = \left\{ f : \int_0^1 \left(\frac{\tilde{f}(y)}{1-y} \right)^*(t) \psi'(t) dt < \infty \right\}.$$

We need to verify that $y \mapsto \|F_y\|_{M(\varphi)} \notin [\mathcal{C}, M(\varphi)]'$; see Proposition 3.5. Since the upper dilation index of φ satisfies $\delta_\varphi < 1$, we can use the equivalent expression (3) for the norm in $M(\varphi)$ to obtain from (10) that

$$\|F_y\|_{M(\varphi)} \asymp \sup_{0 \leq s \leq 1-y} \frac{\varphi(s)}{s+y}.$$

This function is decreasing (as a function of its variable y) and so it coincides with its decreasing majorant, $(\|F_y\|_{M(\varphi)})^{\sim} = \|F_y\|_{M(\varphi)}$. Moreover, for each $y \in [0, 1]$, we have

$$\|F_y\|_{M(\varphi)} \asymp \sup_{0 \leq s \leq 1-y} \frac{\varphi(s)}{s+y} \geq \varphi(1-y),$$

and hence, modulo a positive constant,

$$\frac{\|F_y\|_{M(\varphi)}}{1-y} \geq \frac{\varphi(1-y)}{1-y}.$$

Since φ is quasi-concave, $\varphi(t)/t$ is decreasing and so $(\frac{\varphi(1-t)}{1-y})^*(t) = \varphi(t)/t$, i.e.,

$$\left(\frac{\|F_y\|_{M(\varphi)}}{1-y} \right)^*(t) \geq \frac{\varphi(t)}{t} = \frac{1}{\psi(t)}.$$

Accordingly, modulo a positive constant, we have

$$\int_0^1 \left(\frac{(\|F_y\|_{M(\varphi)})^{\sim}}{1-y} \right)^*(t) \psi'(t) dt \geq \int_0^1 \frac{\psi'(t)}{\psi(t)} dt = \infty.$$

Hence, $\|F_y\|_{\Lambda(\varphi)} \notin [\mathcal{C}, M(\varphi)]'$ and so $L^1(m_{M(\varphi)}) \neq L^1(|m_{M(\varphi)}|)$. Consequently, $[\mathcal{C}, M(\varphi)]$ is not order isomorphic to an AL-space. \square

A precise description of when $[\mathcal{C}, X]$ is a weighted L^1 -space (in particular, an AL-space) can be deduced from [27, Theorem 3.3].

Remark 3.7. (a) The argument at the beginning of the proof of (a) \Rightarrow (b) in Proposition 3.5 shows that also $[\mathcal{C}, M(\varphi)_0]$ is not order isomorphic to an AL-space.

(b) If $[\mathcal{C}, X]$ is order isomorphic to an AL-space, then Proposition 3.5 implies that $[\mathcal{C}, X] = L^1(m_X) \simeq L^1(|m_X|)$. Thus, $\chi_{[0,1]} \in L^1(|m_X|)$ and so m_X has finite variation. Hence, whenever m_X has infinite variation (e.g. $X = L^p_{\text{exp}}$, $p \geq 1$, or if $\log(1/t) \notin \Lambda(\varphi)$), then $[\mathcal{C}, X]$ cannot be order isomorphic to an AL-space.

(c) Further examples of when $[\mathcal{C}, X]$ fails to be order isomorphic to an AL-space occur in Proposition 4.5 below.

The final results of this section address the question of when is X contained in $L^1(m_X)$ or in $L^1(|m_X|)$. In the first case, we have the integral representation for $\mathcal{C}: X \rightarrow X$ as given in (1) via the Bartle-Dunford-Schwartz integral. In the latter case, the representation for $\mathcal{C}: X \rightarrow X$ is via the Bochner integral as given by (2) and (5).

Remark 3.8. (a) Let $X \neq L^\infty$ be a r.i. space such that $\overline{\alpha}_X < 1$. Then each of the containments $X \subseteq [\mathcal{C}, X]$ and $X_a \subseteq L^1(m_X)$ is proper. Indeed, since $\overline{\alpha}_X < 1$, we have $X \subseteq [\mathcal{C}, X]$, where X is r.i. and $[\mathcal{C}, X]$ is not; see Theorem 3.3(b). Thus, $X = [\mathcal{C}, X]$ is impossible.

Applying the previous argument to X_a (in place of X) shows that $X_a \subseteq [\mathcal{C}, X_a]$ properly. But, $[\mathcal{C}, X_a] = L^1(m_X)$; see (11).

If, in addition, X has a.c. norm, then $X_a = X$ and so $X \subseteq L^1(m_X) = [\mathcal{C}, X]$ properly.

(b) Unlike for the containment $X_a \subseteq L^1(m_X)$, it is not true in general (with $\overline{\alpha}_X < 1$) that $X \subseteq L^1(m_X)$. Indeed, for $X = L^{p,\infty}$, $1 < p < \infty$, we have

$$X_a = L^{p,\infty}_0 = \left\{ f : \lim_{t \rightarrow 0} t^{-1/q} \int_0^t f^*(s) ds = 0 \right\}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Since $\overline{\alpha}_X = 1/p < 1$, it follows from part (a) that $L^{p,\infty}_0 \subseteq L^1(m_{L^{p,\infty}})$ properly. To see that $L^{p,\infty} \not\subseteq L^1(m_{L^{p,\infty}})$ we consider, as in the proof of Proposition 3.2, the decreasing function $x^{-1/p} \in L^{p,\infty} \setminus L^{p,\infty}_0$. Since $\mathcal{C}(x^{-1/p}) = qx^{-1/p}$, it follows that $x^{-1/p} \notin [\mathcal{C}, L^{p,\infty}_0]$. From (11) we have $[\mathcal{C}, L^{p,\infty}_0] = L^1(m_{L^{p,\infty}})$. Accordingly, $x^{-1/p} \in L^{p,\infty}$ and $x^{-1/p} \notin L^1(m_{L^{p,\infty}})$.

Let $X = \Lambda(\varphi)$ satisfy $0 < \gamma_\varphi \leq \delta_\varphi < 1$. It follows from Proposition 3.6(a) that $[\mathcal{C}, X] \simeq L^1(|m_X|)$. Since $\overline{\alpha}_X = \delta_\varphi < 1$, we also have $X \subseteq [\mathcal{C}, X]$ and hence, $X \subseteq L^1(|m_X|)$. On the other hand, for $X = L^1$ we have the contrary situation that $L^1 \not\subseteq L^1(\log(1/t)) = L^1(m_{L^1})$; see (12). Of course, here $\overline{\alpha}_X = \delta_\varphi = 1$. A similar situation occurs for $X = L^p$, $1 < p < \infty$, namely $L^p \not\subseteq L^1(|m_{L^p}|)$, [25, Theorem 1.1(ii)]. The following result exhibits additional facts concerning whether or not we have $X \subseteq L^1(|m_X|)$.

Proposition 3.9. *Let $X \neq L^\infty$ be a r.i. space.*

- (a) *It is always the case that $L^1 \not\subseteq L^1(|m_X|)$.*
- (b) *Suppose that $X \subseteq L^1(|m_X|)$. Then the containment is necessarily proper.*

(c) The containment $X \subseteq L^1(|m_X|)$ holds if and only if the function

$$y \mapsto \|F_y\|_X = \left\| t \mapsto \frac{1}{t+y} \chi_{[0,1-y]} \right\|_X, \quad y \in (0, 1],$$

belongs to the associate space X' of X .

(d) For $X = M(\varphi)$, it is the case that $M(\varphi) \not\subseteq L^1(|m_{M(\varphi)}|)$.

Proof. (a) If $L^1 \subseteq L^1(|m_X|)$ holds, then $\int_0^1 |f(y)| \cdot \|F_y\|_X dy < \infty$ for all $f \in L^1$ and so $\sup_{0 < y \leq 1} \|F_y\|_X < \infty$. However, this is impossible since, for each $y \in (0, 1]$, we have

$$\|F_y\|_X \geq \|F_y\|_{L^1} = \int_0^1 \frac{1}{x} \chi_{[y,1]}(x) dx = \log(1/y).$$

(b) If $X \simeq L^1(|m_X|)$ holds, then X is a r.i. space which is order isomorphic to an AL-space. Then, for some constants $C_1, C_2 > 0$, we have $C_1 \|f\|_{L^1(|m_X|)} \leq \|f\|_X \leq C_2 \|f\|_{L^1(|m_X|)}$, $f \in X$. So, for $0 \leq s < t \leq 1$, we have

$$\begin{aligned} \|\chi_{[0,t]}\|_X &\leq C_2 \|\chi_{[0,t]}\|_{L^1(|m_X|)} = C_2 (\|\chi_{[0,s]}\|_{L^1(|m_X|)} + \|\chi_{[s,t]}\|_{L^1(|m_X|)}) \\ &\leq \frac{C_2}{C_1} (\|\chi_{[0,s]}\|_X + \|\chi_{[s,t]}\|_X) = \frac{C_2}{C_1} (\|\chi_{[0,s]}\|_X + \|\chi_{[0,t-s]}\|_X). \end{aligned}$$

In a similar way we can obtain the corresponding lower bound. It follows that the fundamental function φ_X satisfies $\varphi_X(s+t) \asymp \varphi_X(s) + \varphi_X(t)$ for $s, t, s+t \in [0, 1]$. This, together with the continuity of φ_X on $[0, 1]$ and $\varphi_X(0) = 0$, implies that $\varphi_X(ta) \asymp t\varphi_X(a)$ for $t, a, ta \in [0, 1]$. Hence, $\varphi_X(t) \asymp t$ for $t \in [0, 1]$, which implies that X is order isomorphic to L^1 . But, this contradicts part (a).

(c) Note that $X \subseteq L^1(|m_X|)$ if and only if $f \in L^1(|m_X|)$ for all $f \in X$, that is (via Theorem 2.1),

$$\int_0^1 |f(y)| \cdot \|F_y\|_X dy < \infty, \quad f \in X,$$

which corresponds to the function $y \mapsto \|F_y\|_X$ belonging to the space X' . Since X is r.i., it follows from (10) that this is equivalent to the function

$$y \mapsto \|(F_y)^*\|_X = \left\| t \mapsto \frac{1}{t+y} \chi_{[0,1-y]} \right\|_X, \quad y \in (0, 1],$$

belonging to X' .

(d) Applying part (c) to $X = M(\varphi)$ we need to show that $y \mapsto \|F_y\|_{M(\varphi)}$ does not belong to $M(\varphi)' = \Lambda(\psi)$, for $\psi(t) := t/\varphi(t)$.

The function $y \mapsto \|F_y\|_{M(\varphi)}$ can be estimated below, using (10), for the values $0 \leq y \leq 1/2$ (in which case $y \leq 1-y$), namely

$$\begin{aligned} \|F_y\|_{M(\varphi)} &= \sup_{0 < t \leq 1} \frac{\varphi(t)}{t} \int_0^t (F_y)^*(s) ds \\ &\geq \sup_{0 < t \leq 1} \varphi(t) (F_y)^*(t) = \sup_{0 < t \leq 1-y} \frac{\varphi(t)}{y+t} \geq \frac{\varphi(y)}{2y}. \end{aligned}$$

Hence, we have that

$$(\|F_y\|_{M(\varphi)})^*(t) \geq \frac{\varphi(t)}{2t} = \frac{1}{2} \frac{1}{\psi(t)}, \quad 0 < t \leq \frac{1}{2}.$$

Consequently,

$$\|y \mapsto \|F_y\|_{M(\varphi)}\|_{\Lambda(\psi)} = \int_0^1 (\|F_t\|_{M(\varphi)})^*(t) \psi'(t) dt \geq \frac{1}{2} \int_0^{1/2} \frac{\psi'(t)}{\psi(t)} dt = \infty.$$

□

Remark 3.10. The proof of Proposition 3.9(d) shows that the result also applies to the a.c. part $M(\varphi)_0$ of $M(\varphi)$. More generally, for a r.i. space $X \neq L^\infty$ we have $X_a \not\subset L^1(|m_{X_a}|)$ if and only if $X \not\subset L^1(|m_X|)$ since X_a and X have the same norm and $X'_a = X'$.

Regarding the separability of $[\mathcal{C}, X]$, it is known that the B.f.s.' $Ces_p([0, 1])$, $1 < p < \infty$, are *separable*, [2, Theorem 1], [3, Theorem 3.1(b)]. As pointed out in [3, p.18], this is due to the fact that $Ces_p([0, 1])$, which coincides with $[\mathcal{C}, L^p]_a = [\mathcal{C}, L^p]$, contains L^∞ and has a.c. norm. More generally, since $L^\infty \subseteq X_a$ for any r.i. space $X \neq L^\infty$ and $\mathcal{C}: L^\infty \rightarrow L^\infty$, we necessarily have that $L^\infty \subseteq [\mathcal{C}, X_a] = [\mathcal{C}, X]_a$ (cf. Proposition 3.1) and hence, $[\mathcal{C}, X]_a$ is separable by the a.c. of its norm. In particular, if X itself has a.c. norm, then $[\mathcal{C}, X]$ is separable; see (11). Since the σ -algebra \mathcal{M} is λ -essentially countably generated and $L^1(m_X) = [\mathcal{C}, X]_a$, via (11), this also follows from a general result on the separability of $L^1(m)$, [24, Proposition 2].

4. THE CESÀRO OPERATOR ACTING ON $[\mathcal{C}, X]$

It is known that the operator $\mathcal{C}: L^p \rightarrow L^p$, for $1 < p < \infty$, is not compact, [16, p.28]. Actually, this is a rather general feature.

Proposition 4.1. *Let $X \neq L^\infty$ be a r.i. space satisfying $\bar{\alpha}_X < 1$. Then the continuous operator $\mathcal{C}: X \rightarrow X$ is not compact.*

Proof. For each $\alpha \geq 0$, direct calculation shows that the continuous function x^α (on $[0, 1]$) satisfies $\mathcal{C}(x^\alpha) = x^\alpha/(\alpha+1)$ and so $1/(\alpha+1)$ is an eigenvalue of \mathcal{C} . Accordingly, the interval $(0, 1]$ is contained in the spectrum of \mathcal{C} and so \mathcal{C} cannot be compact. □

Since the operator $\mathcal{C}: X \rightarrow X$, whenever it is available, factorizes through $L^1(m_X)$ via $I_{m_X}: L^1(m_X) \rightarrow X$, it follows that also I_{m_X} is not compact. By the same argument also $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$ fails to be compact. Actually, the requirement that $\mathcal{C}: X \rightarrow X$ is unnecessary.

Proposition 4.2. *Let $X \neq L^\infty$ be any r.i. space. Then the operator $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$ is not compact.*

Proof. According to [21, Theorem 4], the bounded variation of m_X is a necessary condition for $I_{m_X}: L^1(m_X) \rightarrow X$ to be compact. Thus, if m_X has infinite variation, then $I_{m_X}: L^1(m_X) \rightarrow X$ is not compact. Since the restriction of $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$ to the closed subspace $L^1(m_X)$ is I_{m_X} , also $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$ fails to be compact.

Suppose now that m_X has finite variation. Then a further condition is necessary for $I_{m_X} : L^1(m_X) \rightarrow X$ to be compact: the existence of a Bochner integrable density, in our case the function $F : y \mapsto F_y$, with the property that the set $\mathcal{B} := \{G(y) := F_y / \|F_y\|_X, 0 \leq y \leq 1\}$ is relatively compact in X , [21, Theorem 1]. So, assume then that this last condition holds. Choose a sequence $\{y_n\} \subseteq [0, 1]$ which increases to 1. Since $\{G_{y_n}\} \subseteq \mathcal{B}$, there is a subsequence, again denoted by $\{G_{y_n}\}$ for convenience, which converges in X . Let $\psi \in X$ be the limit of $\{G_{y_n}\}$. By passing to a subsequence, if necessary, we can assume that $G_{y_n}(x) \rightarrow \psi(x)$ for a.e. $x \in [0, 1]$. Recall that F_y is given by $F_y(x) = (1/x)\chi_{[y,1]}(x)$; see (5). Thus, as $\{y_n\}$ increases to 1 we have $F_{y_n}(x) \rightarrow 0$ for a.e. $x \in [0, 1]$. The same property occurs also for $\{G_{y_n}\}$. As a consequence, $\psi = 0$ a.e. This contradicts $\|\psi\|_X = 1$ as $\|G_{y_n}\|_X = 1$ for all $n \geq 1$. \square

A useful substitution for compactness is the *complete continuity* of an operator, that is, one which maps weakly convergent sequences to norm convergent sequences. In view of the Eberlein-Šmulian Theorem, this is equivalent to mapping relatively weakly compact sets to relatively norm compact sets. For the particular case of the Cesàro operator, due to the fact that the vector measure m_X has relatively compact range and σ -finite variation (cf. Theorem 2.1(c), (d)) it is the case that the (restricted) integration operator $I_{m_X} : L^1(|m_X|) \rightarrow X$ is always completely continuous, [23, Proposition 3.56]. This fact will have important consequences.

The following result should be compared with Proposition 4.1.

Proposition 4.3. *Let $X \neq L^\infty$ be a r.i. space such that the function $y \mapsto \|F_y\|_X$ belongs to X' . Then $\mathcal{C} : X \rightarrow X$ is completely continuous.*

In particular, this occurs for $X = \Lambda(\varphi)$ if φ satisfies $0 < \gamma_\varphi \leq \delta_\varphi < 1$.

Proof. By Proposition 3.9(c) the function $y \mapsto \|F_y\|_X$ belonging to X' implies that $X \subseteq L^1(|m_X|)$. According to (11) we have $X \subseteq [\mathcal{C}, X]$ and so the operator $\mathcal{C} : X \rightarrow X$ is continuous. Moreover, it can be factorized via the continuous inclusion $X \subseteq L^1(|m_X|)$ and the restricted integration operator $I_{m_X} : L^1(|m_X|) \rightarrow X$. But, as noted above, $I_{m_X} : L^1(|m_X|) \rightarrow X$ is necessarily completely continuous. The ideal property of completely continuous operators then implies that $\mathcal{C} : X \rightarrow X$ is also completely continuous.

The particular case of $X = \Lambda(\varphi)$ with $0 < \gamma_\varphi \leq \delta_\varphi < 1$ follows from Proposition 3.5 and Theorem 3.6(a). \square

Remark 4.4. Any r.i. space X for which the function $y \mapsto \|F_y\|_X$ belongs to X' cannot be reflexive. For, if so, then $\mathcal{C} : X \rightarrow X$ is a completely continuous operator defined on a reflexive Banach space and hence, it is necessarily compact (which contradicts Proposition 4.1). For $X = L^p$, $1 < p < \infty$, this was shown explicitly in (the proof of) Theorem 1.1(i) in [25].

We deduce some further consequences from the complete continuity of the restricted integration operator $I_{m_X} : L^1(|m_X|) \rightarrow X$.

Proposition 4.5. *Let $X \neq L^\infty$ be a r.i. space.*

- (a) *If $[\mathcal{C}, X]$ is order isomorphic to an AL-space, then $\mathcal{C} : [\mathcal{C}, X] \rightarrow X$ is completely continuous.*

- (b) Let X be reflexive and satisfy $\overline{\alpha}_X < 1$. Then $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$ is not completely continuous. In particular, $[\mathcal{C}, X]$ cannot be order isomorphic to an AL-space.
- (c) Suppose that X does not contain a copy of ℓ^1 . If $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$ is completely continuous, then $[\mathcal{C}, X]$ is order isomorphic to an AL-space.

Proof. (a) From Proposition 3.5 we have $[\mathcal{C}, X] \simeq L^1(|m_X|)$ which, together with the operator $I_{m_X}: L^1(|m_X|) \rightarrow X$ being completely continuous, establishes the claim.

(b) From $\overline{\alpha}_X < 1$ we have $\mathcal{C}: X \rightarrow X$ and so $X \subseteq [\mathcal{C}, X]$. Then, $\mathcal{C}: X \rightarrow X$ can be factorized via $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$. Suppose that $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$ is completely continuous. Then also $\mathcal{C}: X \rightarrow X$ is completely continuous. Since X is reflexive, we conclude that $\mathcal{C}: X \rightarrow X$ is compact, which is a contradiction to Proposition 4.1.

Suppose now that $[\mathcal{C}, X]$ is order isomorphic to an AL-space. Then part (a) implies that $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$ is completely continuous. But, we have just proved that this is not possible.

(c) Suppose that $[\mathcal{C}, X]$ is not order isomorphic to an AL-space. Then it follows from Proposition 3.5 that $L^1(|m_X|) \neq L^1(m_X)$. On the other hand, the complete continuity of $\mathcal{C}: [\mathcal{C}, X] \rightarrow X$ implies (via factorization through $L^1(m_X)$) that $I_{m_X}: L^1(m_X) \rightarrow X$ is also completely continuous. Combining Corollary 1.4 of [6] with Proposition 1.1 of [22], it follows that $L^1(|m_X|) \simeq L^1(m_X)$. Contradiction! \square

Remark 4.6. For an example of a reflexive r.i. space with $\overline{\alpha}_X = 1$ we refer to [20, Example 12, p.29].

Recall that a Banach space X has the *Dunford-Pettis property* if every Banach-space-valued, weakly compact linear operator defined on X is completely continuous. The classical example of a space with this property is L^1 . In Theorem 3.6(a) it was established, for certain Lorentz spaces $\Lambda(\varphi)$, that $[\mathcal{C}, \Lambda(\varphi)] \simeq L^1(|m_{\Lambda(\varphi)}|)$ with $|m_{\Lambda(\varphi)}|$ a finite, non-atomic measure. Hence, $[\mathcal{C}, \Lambda(\varphi)]$ has the Dunford-Pettis property in this case. However, as noted in the Introduction, $Ces_p = [\mathcal{C}, L^p]$, $1 < p < \infty$, fails the Dunford-Pettis property, [2, §6, Corollary 1]. The proof of this given in [2] relies on some results concerning certain Banach space properties particular to Ces_p . The following extension of this result is established via the methods of vector measures.

Proposition 4.7. *Let X be any reflexive r.i. space with $\overline{\alpha}_X < 1$. Then, $[\mathcal{C}, X]$ fails the Dunford-Pettis property.*

Proof. Since X has a.c. norm, we have $L^1(m_X) = [\mathcal{C}, X]$ (cf. (11)) and hence, because of $\overline{\alpha}_X < 1$, it follows that $\mathcal{C}: X \rightarrow X$ and so $X \subseteq [\mathcal{C}, X] = L^1(m_X)$. Suppose that $[\mathcal{C}, X]$ has the Dunford-Pettis property. Then the weakly compact operator $I_{m_X}: L^1(m_X) \rightarrow X$ (recall that X is reflexive) is necessarily completely continuous. Since $\mathcal{C}: X \rightarrow X$ is the composition of $I_{m_X}: L^1(m_X) \rightarrow X$ and the natural inclusion of X into $L^1(m_X)$, it follows that $\mathcal{C}: X \rightarrow X$ is completely continuous. The reflexivity of X then ensures that $\mathcal{C}: X \rightarrow X$ is actually compact. But, this contradicts Proposition 4.1. Accordingly, $[\mathcal{C}, X]$ fails the Dunford-Pettis property. \square

5. THE FATOU PROPERTY FOR $[\mathcal{C}, X]$

In [17, Theorem 1(d)] it was noted that if X has the Fatou property, then also $[\mathcal{C}, X]$ has the Fatou property. As explained in the beginning of §3, the results on optimal domains for kernel operators given in [13, §3] also apply to the kernel generating the Cesàro operator (and to many other operators). In [13], a fine analysis of the Fatou property was undertaken. Proposition 3.1 above presents a partial view of the relations between the various function spaces involved. The complete picture of the results in [13, §3] is presented below. It involves the space $L_w^1(m_X)$ consisting of all the functions which are *weakly integrable* with respect to the vector measure m_X , that is, of all measurable functions $f: [0, 1] \rightarrow \mathbb{R}$ such that $f \in L^1(|x^*m_X|)$, for every $x^* \in X^*$. It is a B.f.s. for the “same” norm (4) as used in $L^1(m_X)$ and contains $L^1(m_X)$ as a closed subspace, [23, Ch.3, §1]. The Copson operator \mathcal{C}^* was defined in the proof of Theorem 3.6. Whenever X has a.c. norm and $\overline{\alpha}_X < 1$ it is the dual operator to $\mathcal{C}: X \rightarrow X$.

The following result is a summary of facts that occur in [13], specialized to the Cesàro operator. Parts (a), (f) already occur in Proposition 3.1 and (b) also occurs in [17, Theorem 1]. Part (k) is Theorem 3.1 of [27]; it provides an alternate description of $[\mathcal{C}, X]'$ to that given in (13).

Proposition 5.1. *Let $X \neq L^\infty$ be a r.i. space.*

- (a) *If X has a.c. norm, then $[\mathcal{C}, X]$ has a.c. norm and $[\mathcal{C}, X] = L^1(m_X)$.*
- (b) *If X has the Fatou property, then $[\mathcal{C}, X]$ has the Fatou property.*
- (c) *If X has the weak Fatou property, then $[\mathcal{C}, X]$ has the weak Fatou property.*
- (d) *If X' is a norming subspace of X^* , then $[\mathcal{C}, X]'$ is a norming subspace of $[\mathcal{C}, X]^*$.*
- (e) *If X' is a norming subspace of X^* , then $[\mathcal{C}, X]'' = [\mathcal{C}, X'']$.*
- (f) *If $f \in L^1(m_X)$, then $f \in [\mathcal{C}, X]$ and $\|f\|_{L^1(m_X)} = \|f\|_{[\mathcal{C}, X]}$.*
- (g) *If $f \in [\mathcal{C}, X]$, then $f \in L_w^1(m_X)$ and $\|f\|_{L_w^1(m_X)} \leq \|f\|_{[\mathcal{C}, X]}$.*
- (h) *If $f \in L_w^1(m_X)$, then $f \in [\mathcal{C}, X'']$ and $\|f\|_{[\mathcal{C}, X'']} \leq \|f\|_{L_w^1(m_X)}$.*
- (i) *$[\mathcal{C}, X]'' = L_w^1(m_X)$ with equality of norms.*
- (j) *If X' is a norming subspace of X^* , then $L_w^1(m_X) = [\mathcal{C}, X'']$.*
- (k) *If X has a.c. norm, the Fatou property and satisfies $\overline{\alpha}_X < 1$, then $[\mathcal{C}, X]'$ equals the ideal in L^0 generated by the range $\{\mathcal{C}^*(f) : f \in X'\}$ where \mathcal{C}^* acts in X' .*

In the event that X' is a norming subspace of X^ , there is equality of norms in (g) and (h).*

The following chain of inclusions, which refines (11), summarizes the situation (cf. (9) on p.199 of [13]):

$$(14) \quad L^1(m_X) \subseteq [\mathcal{C}, X] \subseteq [\mathcal{C}, X]'' = L_w^1(m_X) = L^1(m_X)'' \subseteq [\mathcal{C}, X''].$$

If X has a.c. norm, then the first and last containments are equalities and the second containment an isometric embedding. On the other hand, if X has the Fatou property (i.e., $X = X''$), then the second and last containments are equalities. Finally, in case X has both a.c. norm and the Fatou property (i.e., X is weakly sequentially complete), then all spaces involved coincide.

It should be stressed that the space $L_w^1(m_X)$ plays a crucial role. Recall that whenever a B.f.s. X space does not have the Fatou property, then it is always possible to identify its ‘Fatou completion’, that is, the smallest of all B.f.s. which contain X and have the Fatou property, [29, §71, Theorem 2]. This space coincides with X'' . Proposition 5.1(i) shows that the space $L_w^1(m_X)$ is precisely the Fatou completion of the Cesàro space $[\mathcal{C}, X]$, whereas $L^1(m_X)$ is the a.c. part of $[\mathcal{C}, X]$.

We conclude with two relevant results. Recall (cf. Theorem 3.3(a)) that the space $[\mathcal{C}, X]$ is never reflexive. Weak sequential completeness of a B.f.s. is often a good replacement for the space failing to be reflexive.

Proposition 5.2. *Let $X \neq L^\infty$ be a r.i space.*

- (a) *If the integration operator $I_{m_X}: L^1(m_X) \rightarrow X$ is weakly compact, then $[\mathcal{C}, X]$ is weakly sequentially complete.
If, in addition, $\overline{\alpha}_X < 1$, then $\mathcal{C}: X \rightarrow X$ is also weakly compact.*
- (b) *If the integration operator $I_{m_X}: L^1(m_X) \rightarrow X$ is completely continuous, then $[\mathcal{C}, X]$ is weakly sequentially complete.
If, in addition, $\overline{\alpha}_X < 1$, then $\mathcal{C}: X \rightarrow X$ is also completely continuous.*

Proof. (a) If $I_{m_X}: L^1(m_X) \rightarrow X$ is weakly compact, then Corollary 2.3 of [14] asserts that $L^1(m_X) = L_w^1(m_X)$ and hence, $L^1(m_X)$ has the Fatou property; see (14). Being also a.c., it follows that $L^1(m_X)$ is weakly sequentially complete. Again according to (14) we then have $L^1(m_X) = [\mathcal{C}, X] = L_w^1(m_X)$.

If, in addition, $\mathcal{C}: X \rightarrow X$, then \mathcal{C} factorizes through $L^1(m_X)$ via I_{m_X} and so is itself also weakly compact.

(b) If $I_{m_X}: L^1(m_X) \rightarrow X$ is completely continuous, then again it is known that necessarily $L^1(m_X) = L_w^1(m_X)$, [7, Theorem 3.6]. A similar argument as in the proof of (a) establishes the result. \square

REFERENCES

1. S.V. Astashkin and L. Maligranda, *Cesàro function spaces fail the fixed point property*, Proc. Amer. Math. Soc. **136** (2008), 4289–4294.
2. S.V. Astashkin and L. Maligranda, *Structure of Cesàro function spaces*, Indag. Math. (N.S.) **20** (2009), 329–379.
3. S.V. Astashkin and L. Maligranda, *Structure of Cesàro function spaces: a survey*. In: Function spaces X, Banach Center Publ., **102**, Polish Acad. Sci. Inst. Math., Warsaw, (2014), 13–40.
4. R. G. Bartle, N. Dunford and J. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. **7** (1955), 289–305.
5. C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press Inc., Boston (1988).
6. J. M. Calabuig, J. Rodríguez and E. A. Sánchez-Pérez, *On completely continuous integration operators of a vector measure*, J. Convex Anal., **21** (2014), 811–818.
7. R. del Campo, S. Okada and W. J. Ricker, *L^p -spaces and ideal properties of integration operators for Fréchet-space-valued measures*, J. Operator Theory **68** (2012), 463–485.
8. G. P. Curbera, *Operators into L^1 of a vector measure and applications to Banach lattices*, Math. Ann. **293** (1992), 317–330.
9. G. P. Curbera, *When L^1 of a vector measure is an AL-space*, Pacific J. Math. **162** (1994), 287–303.
10. G. P. Curbera, *Banach space properties of L^1 of a vector measure*, Proc. Amer. Math. Soc. **123** (1995), 3797–3806.

11. G. P. Curbera, *A note on function spaces generated by Rademacher series*, Proc. Edinburgh Math. Soc. **40** (1997), 119–126.
12. G. P. Curbera and W. J. Ricker, *Optimal domains for kernel operators via interpolation*, Math. Nachr. **244** (2002), 47–63.
13. G. P. Curbera and W. J. Ricker, *Banach lattices with the Fatou property and optimal domains of kernel operators*, Indag. Math. (N. S.), **17** (2006), 187–204.
14. G. P. Curbera, O. Delgado and W. J. Ricker, *Vector measures: where are their integrals?*, Positivity, **13** (2009), 61–87.
15. S. G. Krein, Ju. I. Petunin and E. M. Semenov, *Interpolation of Linear Operators*, Amer. Math. Soc., Providence, (1982).
16. G.M. Leibowitz, *Spectra of finite range Cesàro operators*, Acta Sci. Math. (Szeged), **35** (1973), 27–29.
17. K. Leśnik and M. Maligranda, *On abstract Cesàro spaces. Duality*, J. Math. Anal. Appl., **424** (2015), 932–951.
18. K. Leśnik and M. Maligranda, *On abstract Cesàro spaces. Optimal range*, Integral Equations Operator Theory, **81** (2015), 227–235.
19. J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces* vol. II, Springer-Verlag, Berlin, (1979).
20. L. Maligranda, *Indices and interpolation*, Dissert. Math. **234** (1985), 1–54.
21. S. Okada, W. J. Ricker and L. Rodríguez-Piazza, *Compactness of the integration operator associated with a vector measure*, Studia Math. **150** (2002), 133–149.
22. S. Okada, W. J. Ricker and L. Rodríguez-Piazza, *Operator ideal properties of vector measures with finite variation*, Studia Math. **205** (2011), 215–249.
23. S. Okada, W. J. Ricker and E. Sánchez-Pérez, *Optimal Domain and Integral Extension of Operators acting in Function Spaces*, Operator Theory Advances Applications **180**, Birkhäuser Verlag, Basel-Berlin-Boston, (2008).
24. W.J. Ricker, *Separability of the L^1 -space of a vector measure*, Glasgow Math. J., **34** (1992), 1–9.
25. W.J. Ricker, *Optimal extension of the Cesàro operator in $L^p([0, 1])$* , Bull. Belg. Math. Soc. Simon Stevin, **22** (2015), 343–352.
26. V.A. Rodin and E.M. Semenov, *Rademacher series in symmetric spaces*, Anal. Math., **1** (1975), 207–222.
27. A.R. Schep, *When is the optimal domain of a positive linear operator a weighted L^1 -space?*, Vector Measures, Integration and Related Topics, pp.361-369, Oper. Theory Adv. Appl., vol. 201, Birkhäuser, Basel, 2010.
28. D. van Dulst, *Characterizations of Banach spaces not containing ℓ^1* , CWI Tract No. 59, Centrum voor Wiskunde en Informatica, Amsterdam, 1989.
29. A. C. Zaanen, *Integration*, 2nd rev. ed. North Holland, Amsterdam; Interscience, New York Berlin (1967).

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